Unit 2 Matrices

Day #1 Modeling with Matrices

Objective: You will add, subtract, and find the produc of two or more matrices.

Matrix

$$\begin{bmatrix} & \\ & \end{bmatrix}$$

- Each term is called an eleme
- Dimensions : $m \times n$
- m: # of rows (horizontal)
- n: # of columns (vertical)

Equal Matrices
- Two matrices are equal if they have the same dimensions and elements.

Ex: Find x and y: $\begin{bmatrix} y-3 \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix}$

$y-3 = x$ $\rightarrow$ $2x - 3 = x$

$y = 2x$ $\quad -2x \quad -2x$

$-3 = -x$

$\boxed{x=3}$

$y = 2(3)$

$\boxed{y=6}$

Adding Matrices

- Must have the same dimensions
- Add the corresponding elements.

Ex: $\begin{bmatrix} 3 & 5 & -7 \\ -1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} -2 & 8 & 6 \\ 5 & -9 & 10 \end{bmatrix}$

$= \begin{bmatrix} 3+-2 & 5+8 & -7+6 \\ -1+5 & 0-9 & 4+10 \end{bmatrix} = \begin{bmatrix} 1 & 13 & -1 \\ 4 & -9 & 14 \end{bmatrix}$

Subtracting Matrices

- Must add the opposite

Ex:
$$\begin{bmatrix} 3 & 5 & -7 \\ -1 & 0 & 4 \end{bmatrix} - \begin{bmatrix} -2 & 8 & 6 \\ 5 & -9 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 5 & -7 \\ -1 & 0 & 4 \end{bmatrix} + (-1)\begin{bmatrix} -2 & 8 & 6 \\ 5 & -9 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 5 & -7 \\ -1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 2 & -8 & -6 \\ -5 & 9 & -10 \end{bmatrix} = \begin{bmatrix} 5 & -3 & -13 \\ -6 & 9 & -6 \end{bmatrix}$$

Scaler Product

- Multiplying a matrix by a constant.

Ex:
$$3\begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(3) \\ 3(-2) & 3(4) \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ -6 & 12 \end{bmatrix}$$

Product of Matrices

The product of an $m \times n$ matrix and an $n \times r$ matrix is an $m \times r$ matrix.

Ex:
$$\begin{array}{cc} a & b \\ \end{array} \qquad \begin{array}{cc} x & y \\ \end{array}$$
$$\begin{bmatrix} 3 & 2 \\ -1 & 0 \\ 4 & 7 \end{bmatrix} \cdot \begin{bmatrix} 4 & -3 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} a_1 x_1 + b_1 x_2 & a_1 y_1 + b_1 y_2 \\ a_2 x_1 + b_2 x_2 & a_2 y_1 + b_2 y_2 \\ a_3 x_1 + b_3 x_2 & a_3 y_1 + b_3 y_2 \end{bmatrix}$$

$3 \times 2 \qquad 2 \times 2$

Same so can multiply

Resultant Matrix is 3×2

$$= \begin{bmatrix} 3(4) + 2(5) & 3(-3) + 2(1) \\ -1(4) + 0(5) & -1(-3) + 0(1) \\ 4(4) + 7(5) & 4(-3) + 7(1) \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -7 \\ -4 & 3 \\ 51 & -5 \end{bmatrix}$$

Ex:

$$\begin{bmatrix} 8 & 3 & -2 \\ 4 & -1 & -5 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 2 \\ -3 & 2 & 6 \\ 4 & 4 & -2 \end{bmatrix}$$

$$\underbrace{2 \times \underbrace{3 \quad\quad 3}_{\text{yes}} \times 3}_{2 \times 3}$$

$$= \begin{bmatrix} 8(0)+3(-3)+-2(4) & 8(1)+3(2)+-2(4) & 8(2)+3(6)+-2(-2) \\ 4(0)+-1(-3)+-5(4) & 4(1)+-1(2)+-5(4) & 4(2)+-1(6)+-5(-2) \end{bmatrix}$$

$$= \begin{bmatrix} -17 & 6 & 20 \\ -17 & -18 & 18 \end{bmatrix}$$

Homework: Pg 83 # 22, 35, 37, 40, 45

Day #2 Determinants

Objective: You will evaluate determinants by minors and the lattice method.

Determinants

Every square matrix has a determinant (2x2, 3x3, 4x4, etc)

The determinant of $\begin{bmatrix} 8 & 7 \\ 4 & 5 \end{bmatrix}$ is written:

$$\begin{vmatrix} 8 & 7 \\ 4 & 5 \end{vmatrix} \text{ or } \det\begin{bmatrix} 8 & 7 \\ 4 & 5 \end{bmatrix}$$

General Rule for 2x2 Determinants

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - cb$$

Ex: Evaluate: $\det\begin{bmatrix} 8 & 7 \\ 4 & 5 \end{bmatrix} = 8(5) - 4(7)$

$= 40 - 28$

$= 12$

Ex: Evaluate: $\begin{vmatrix} -1 & -3 \\ 2 & 5 \end{vmatrix} = -1(5) - (-3)(2)$

$= -5 + 6$

$= 1$

Minor Method

Used for determining 3x3 determinants.

1) Multiply each term in the first row by its minor (cross out the row and column containing the element and evaluate the resulting matrix.)

2) Combine the resulting numbers, alternating between addition and subtracting.

Ex: $\begin{vmatrix} -4 & -6 & 2 \\ 5 & -1 & 3 \\ -2 & 4 & -3 \end{vmatrix}$

$$= -4\begin{vmatrix} -1 & 3 \\ 4 & -3 \end{vmatrix} - (-6)\begin{vmatrix} 5 & 3 \\ -2 & -3 \end{vmatrix} + 2\begin{vmatrix} 5 & -1 \\ -2 & 4 \end{vmatrix}$$

$$= -4\big(-1(-3) - 3(4)\big) + 6\big(5(-3) - 3(-2)\big) + 2\big(5(4) - (-1)(-2)\big)$$

$$= -4(3-12) + 6(-15+6) + 2(20-2)$$

$$= -4(-9) + 6(-9) + 2(18) = 36 - 54 + 36 = 18$$

Lattice Method

Used for determing 3 x 3 determinants

1) Rewrite the first 2 columns of the matrix on the outside of the matrix.
2) Multiply along the diagonals in both directions.
3) When multiplying from left to right, add the terms. When multiplying from right to left, subtract the terms.

Ex: $\left|\begin{matrix} -4 & -6 & 2 \\ 5 & -1 & 3 \\ -2 & 4 & -3 \end{matrix}\right| \begin{matrix} -4 & -6 \\ 5 & -1 \\ -2 & 4 \end{matrix}$

$$= (-4)(-1)(-3) + (-6)(3)(-2) + (2)(5)(4) - (2)(-1)(-2) - (-4)(3)(4) - (-6)(5)(-3)$$

$$= -12 + 36 + 40 - 4 - (-48) - (90)$$

$$= 18$$

Homework: Pg 102

\# 15, 16, 17, 20, 21

Day #3 Multiplicative Inverses

Objective: You will find the mult. inverse of a matrix.

Review: Multiplicative Inverses and Identities.

Mult. Identity: $5 \cdot \underline{1} = 5$

Mult. Inverse: $5 \cdot \underline{1/5} = 1$

Multiplicative Inverse of a Matrix

If $A = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$ then $A^{-1} = \dfrac{1}{\det[A]} \begin{bmatrix} z & -x \\ -y & w \end{bmatrix}$

Example #1: Find the inverse of $A = \begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}$

$$A^{-1} = \frac{1}{\begin{vmatrix} 2 & -3 \\ 4 & 4 \end{vmatrix}} \begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{8-(-12)} \begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{20} \begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{3}{20} \\ -1/5 & \frac{1}{10} \end{bmatrix}$$

Example #2: Find the inverse of $B = \begin{bmatrix} 6 & 3 \\ -1 & 2 \end{bmatrix}$

$$B^{-1} = \frac{1}{\begin{vmatrix} 6 & 3 \\ -1 & 2 \end{vmatrix}} \begin{bmatrix} 2 & -3 \\ 1 & 6 \end{bmatrix}$$

$$B^{-1} = \frac{1}{12-(3)} \begin{bmatrix} 2 & -3 \\ 1 & 6 \end{bmatrix}$$

$$B^{-1} = \frac{1}{15} \begin{bmatrix} 2 & -3 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} \frac{2}{15} & -\frac{1}{5} \\ \frac{1}{15} & \frac{2}{5} \end{bmatrix}$$

Homework:
Pg 103
\# 27, 28, 29

Day #4 Matrix Equations

Objective: You will solve a system of equations by using matrix equations.

Matrix Equations

A method of solving a system of equations using the multiplicative inverse of a matrices.

Recall: $A \cdot A^{-1} = I$

$A^{-1} \cdot A = I$

Example #1 Solve using matrix equations: $x + 5y = 26$
$3x - 2y = -41$

$$\underbrace{\begin{bmatrix} 1 & 5 \\ 3 & -2 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 26 \\ -41 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\begin{vmatrix} 1 & 5 \\ 3 & -2 \end{vmatrix}} \begin{bmatrix} -2 & -5 \\ -3 & 1 \end{bmatrix}$$

now.... $\underbrace{A^{-1} \cdot A}_{I} \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} 26 \\ -41 \end{bmatrix}$

$I \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} 26 \\ -41 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} =$ Answer

$$\underbrace{\frac{1}{\begin{vmatrix} 1 & 5 \\ 3 & -2 \end{vmatrix}} \begin{bmatrix} -2 & -5 \\ -3 & 1 \end{bmatrix}}_{A^{-1}} \cdot \underbrace{\begin{bmatrix} 1 & 5 \\ 3 & -2 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\begin{vmatrix} 1 & 5 \\ 3 & -2 \end{vmatrix}} \begin{bmatrix} -2 & -5 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 26 \\ -41 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-2-15} \begin{bmatrix} -52 + 205 \\ -78 - 41 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-17} \begin{bmatrix} 153 \\ -119 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -9 \\ 7 \end{bmatrix}$$

$\boxed{(-9, 7)}$

Example #2

Solve using matrix equations: $\begin{aligned} 4x - y &= 1 \\ x + 2y &= 7 \end{aligned}$

$$\begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\begin{vmatrix} 4 & -1 \\ 1 & 2 \end{vmatrix}} \cdot \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{8-(-1)} \begin{bmatrix} 2 + 7 \\ -1 + 28 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 9 \\ 27 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \boxed{(1,3)}$$

Homework Pg 103
38, 39

Day #5 Cramer's Rule

<u>Objective:</u> You will solve a system of equations by using Cramer's Rule.

<u>Cramer's Rule</u>

A method of solving systems of equations using determinants.

By substituting the "solution" in for each variable, you can solve for the variable.

<u>Solve using Cramer's Rule:</u>

$$x + 3y = 1$$
$$2x + 8y = -3$$

$$D = \begin{vmatrix} 1 & 3 \\ 2 & 8 \end{vmatrix}$$
$$8 - 6 = 2$$

$$X = \frac{\begin{vmatrix} 1 & 3 \\ -3 & 8 \end{vmatrix}}{2} = \frac{8+9}{2} = \frac{17}{2} = \boxed{8.5}$$

$$Y = \frac{\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}}{2} = \frac{-3-2}{2} = \boxed{\frac{-5}{2}}$$

$$(8.5, -5/2)$$

Example #2: Solve using Cramers Rule

$$2x + 3y = 3$$
$$12x - 15y = -4$$

$$D = \begin{vmatrix} 2 & 3 \\ 12 & -15 \end{vmatrix}$$
$$= -30 - 36$$
$$= -66$$

$$X = \frac{\begin{vmatrix} 3 & 3 \\ -4 & -15 \end{vmatrix}}{-66} = \frac{-45 - (-12)}{-66} = \frac{-33}{-66} = \boxed{\frac{1}{2}}$$

$$Y = \frac{\begin{vmatrix} 2 & 3 \\ 12 & -4 \end{vmatrix}}{-66} = \frac{-8 - (36)}{-66} = \frac{-44}{-66} = \boxed{\frac{2}{3}}$$

$$\left(\frac{1}{2}, \frac{2}{3}\right)$$

Example #3 Solve using Cramers Rule: $3x-2y+3z=11$
$2x+3y-2z=-5$
$x+4y-z=-5$

$$D=\begin{vmatrix} 3 & -2 & 3 \\ 2 & 3 & -2 \\ 1 & 4 & -1 \end{vmatrix}\begin{matrix} 3 & -2 \\ 2 & 3 \\ 1 & 4 \end{matrix}$$

$$= -9+(+4)+24-(+4)-(-24)-9$$
$$= 30$$

$$X=\frac{\begin{vmatrix} 11 & -2 & 3 \\ -5 & 3 & -2 \\ -5 & 4 & -1 \end{vmatrix}}{30} = \frac{11\left(-3-(-8)\right)-(-2)\left(5-10\right)+3\left(-20-(-15)\right)}{30}$$

$$= \frac{11(5)+2(-5)+3(-5)}{30}$$

$$= \frac{55-10-15}{30} = \frac{30}{30} = 1$$

$$Y=\frac{\begin{vmatrix} 3 & 11 & 3 \\ 2 & -5 & -2 \\ 1 & -5 & -1 \end{vmatrix}}{30} = \frac{3(5-10)-11\left(-2-(-2)\right)+3\left(-10-(-5)\right)}{30}$$

$$= \frac{3(-5)-11(0)+3(-5)}{30}$$

$$= \frac{-15-0-15}{30} = \frac{-30}{30} = -1$$

$$Z=\frac{\begin{vmatrix} 3 & -2 & 11 \\ 2 & 3 & -5 \\ 1 & 4 & -5 \end{vmatrix}}{30} = \frac{3\left(-15-(-20)\right)-(-2)\left(-10-(-5)\right)+11(8-3)}{30}$$

$$= \frac{3(5)+2(-5)+11(5)}{30}$$

$$= \frac{15-10+55}{30} = \frac{60}{30} = 2$$

$(1,-1,2)$

Day #6 – Augmented Matrices & Reduced Row-Echelon Fo

Objective – You will.... See the Heading ☺

Augmented Matrix – a matrix composed of the coefficient of each variable and the constant terms (answers)

$2x + 3y = 6$
$-4x - 5y = 1$

Augmented Matrix:

$$\left[\begin{array}{cc|c} 2 & 3 & 6 \\ -4 & -5 & 1 \end{array}\right]$$

may be omitted.

Reduced Row-Echelon Form:

$$\begin{bmatrix} 1 & 0 & C_1 \\ 0 & 1 & C_2 \end{bmatrix} \qquad x = C_1 \quad y = C_2 \qquad (C_1, C_2)$$

$$\begin{bmatrix} 1 & 0 & 0 & C_1 \\ 0 & 1 & 0 & C_2 \\ 0 & 0 & 1 & C_3 \end{bmatrix} \qquad x = C_1 \quad y = C_2 \quad z = C_3 \qquad (C_1, C_2, C_3)$$

Row Operations on Matrices

To get a 1:

- Multiply the row by the reciprocal of the number needing to become 1.

To get a 0:

Multiply the other row by the opposite number and add the rows together.

You can also interchange the rows!! ☺

Example #1

Solve using row operations: $\begin{aligned} 4x - 7y &= -2 \\ x + 2y &= 7 \end{aligned}$

$$R_1 \leftrightarrow R_2 \quad \left[\begin{array}{cc|c} 4 & -7 & -2 \\ 1 & 2 & 7 \end{array}\right] \qquad \text{Want } \left[\begin{array}{cc|c} 1 & 0 & c_1 \\ 0 & 1 & c_2 \end{array}\right]$$

Start

Then

Then

Then

$$-4R_1 + R_2 \to R_2 \quad \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 4 & -7 & -2 \end{array}\right]$$

$$-\frac{1}{15}R_2 \to R_2 \quad \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & -15 & -30 \end{array}\right]$$

$$-2R_2 + R_1 \to R_1 \quad \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right] \qquad \begin{aligned} x &= 3 \\ y &= 2 \end{aligned} \qquad \boxed{(3,2)}$$

Example #2

Solve using row operations: $\begin{aligned} x - 2y &= 1 \\ 3x + 5y &= -8 \end{aligned}$

$$-3R_1 + R_2 \to R_2 \quad \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 3 & 5 & -8 \end{array}\right]$$

$$\frac{1}{11}R_2 \to R_2 \quad \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 11 & -11 \end{array}\right]$$

$$2R_2 + R_1 \to R_1 \quad \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & -1 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -1 \end{array}\right] \qquad \begin{aligned} x &= -1 \\ y &= -1 \end{aligned} \qquad (-1,-1)$$

Homework: Solve using row operations:

1) $\begin{aligned} 3x + 3y &= -9 \\ -2x + y &= -4 \end{aligned}$

2) $\begin{aligned} 4x - 3y &= 1 \\ 3x + 2y &= 5 \end{aligned}$

Day 7 – Row Operations w/ 3 Variables.

Reduced Row-Echelon Form:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \end{array}\right] \qquad \begin{aligned} x &= c_1 \\ y &= c_2 \\ z &= c_3 \end{aligned} \qquad (c_1, c_2, c_3)$$

Solve using row operations:

$$\begin{aligned} 2x + y - 2z &= 7 \\ x - 2y - 5z &= -1 \\ 4x + y + z &= -1 \end{aligned}$$

$\frac{1}{2}R_1 \to R_1$
$$\left[\begin{array}{ccc|c} 2 & 1 & -2 & 7 \\ 1 & -2 & -5 & -1 \\ 4 & 1 & 1 & -1 \end{array}\right]$$

$R_2 - R_1 \to R_2$
$$\left[\begin{array}{ccc|c} 1 & 1/2 & -1 & 3\,1/2 \\ 1 & -2 & -5 & -1 \\ 4 & 1 & 1 & -1 \end{array}\right]$$

$-4R_1 + R_3 \to R_3$
$$\left[\begin{array}{ccc|c} 1 & 1/2 & -1 & 3\,1/2 \\ 0 & -2.5 & -6 & -4.5 \\ 4 & 1 & 1 & -1 \end{array}\right]$$

$2R_1 + R_3 \to R_3$
$$\left[\begin{array}{ccc|c} 1 & 1/2 & -1 & 3\,1/2 \\ 0 & -2.5 & -6 & -4.5 \\ 0 & -1 & 5 & -15 \end{array}\right]$$

$-\frac{2}{5}R_2 \to R_2$
$$\left[\begin{array}{ccc|c} 1 & 1/2 & -1 & 3\,1/2 \\ 0 & -2.5 & -6 & -4.5 \\ 0 & 0 & 3 & -8 \end{array}\right]$$

$-\frac{1}{2}R_2 + R_1 \to R_1$
$$\left[\begin{array}{ccc|c} 1 & 1/2 & -1 & 3\,1/2 \\ 0 & 1 & 2.4 & 1.8 \\ 0 & 0 & 3 & -8 \end{array}\right]$$

$\frac{1}{3}R_3 \to R_3$
$$\left[\begin{array}{ccc|c} 1 & 0 & -2.2 & 2.6 \\ 0 & 1 & 2.4 & 1.8 \\ 0 & 0 & 3 & -8 \end{array}\right]$$

$2.2R_3 + R_1 \to R_1$
$$\left[\begin{array}{ccc|c} 1 & 0 & -2.2 & 2.6 \\ 0 & 1 & 2.4 & 1.8 \\ 0 & 0 & 1 & -8/3 \end{array}\right]$$

$-2.4R_3 + R_2 \to R_2$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -49/15 \\ 0 & 1 & 2.4 & 1.8 \\ 0 & 0 & 1 & -8/3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -49/15 \\ 0 & 1 & 0 & 8.2 \\ 0 & 0 & 1 & -8/3 \end{array}\right]$$

$x = -49/15$
$y = 8.2$
$z = -8/3$

$$\left(-\frac{49}{15},\ 8.2,\ -8/3\right)$$

HW:

$$\begin{aligned} x + y + z &= 6 \\ 2x - 3y + 4z &= 3 \\ 4x - 8y + 4z &= 12 \end{aligned}$$